

# A LIMIT THEOREM FOR DISCRETE QUANTUM GROUPS

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ABSTRACT. We consider the *concentration functions problem* for discrete quantum groups; we prove that if  $\mathbb{G}$  is a discrete quantum group, and  $\mu$  is an irreducible state in  $l^1(\mathbb{G})$ , then the convolution powers  $\mu^n$ , considered as completely positive maps on  $c_0(\mathbb{G})$ , converge to zero in strong operator topology.

Consider an irreducible probability measure  $\mu$  on a discrete group  $G$  (that is,  $G$  is generated as a semi-group, by the support of  $\mu$ ); then under what conditions on  $G$  or  $\mu$ , does the sequence

$$(0.1) \quad f_n(K) = \sup \{ \mu^n(Kx^{-1}) : x \in G \}$$

converges to zero for every finite set  $K \subset G$ ? This problem is known as the *concentration functions problem*. In [1], Hofmann–Mukherjea considered, and partially answered this problem for the case of locally compact groups. In particular they proved that the sequence (0.1) converges to zero for all discrete  $G$ , finite  $K \subseteq G$ , and irreducible  $\mu \in l^1(G)$ . It is not hard to see that the convergence of the concentration functions to zero is equivalent to the convergence of the convolution operators  $\mu^n$  on  $c_0(\mathbb{G})$ , to zero in the strong operator topology.

Here we prove a non-commutative version of this well-known classical result for the case of discrete quantum groups (Theorem 2).

We recall that a *discrete quantum group*  $\mathbb{G}$  (in the sense of [3]) is a quadruple  $(l^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ , where  $\Gamma : l^\infty(\mathbb{G}) \rightarrow l^\infty(\mathbb{G}) \bar{\otimes} l^\infty(\mathbb{G})$  is a co-associative co-multiplication on the von Neumann algebra  $l^\infty(\mathbb{G})$ , and  $\varphi$  and  $\psi$  are (normal faithful semi-finite) left and right Haar weights on  $l^\infty(\mathbb{G})$ , respectively.

The preadjoint of the co-multiplication  $\Gamma$  induces an associative multiplication on  $l^1(\mathbb{G}) := l^\infty(\mathbb{G})_*$  given by the convolution

$$\mu \star \nu = (\mu \otimes \nu) \Gamma.$$

With this product  $l^1(\mathbb{G})$  is a completely contractive unital Banach algebra. The convolution actions  $x \star \mu := (\iota \otimes \mu) \Gamma(x)$  and  $\mu \star x := (\mu \otimes \iota) \Gamma(x)$  are normal completely bounded maps on  $l^\infty(\mathbb{G})$ .

The *reduced quantum group  $C^*$ -algebra* associated to  $\mathbb{G}$  is denoted by  $c_0(\mathbb{G})$ , which is a weak\* dense  $C^*$ -subalgebra of  $l^\infty(\mathbb{G})$ . Similar to the classical case, we have the duality  $l^1(\mathbb{G}) = c_0(\mathbb{G})^*$ .

We denote by  $\mathcal{P}(\mathbb{G})$  the set of all ‘quantum probability measures’ on  $\mathbb{G}$  (i.e. all normal states in  $l^1(\mathbb{G})$ ). For any such element the convolution action is a *Markov operator*, i.e. a unital normal completely positive map, on  $l^\infty(\mathbb{G})$ .

A state  $\mu \in \mathcal{P}(\mathbb{G})$  is *irreducible* if for every non-zero element  $x \in l^\infty(\mathbb{G})^+$  there exists  $n \in \mathbb{N}$  such that  $\langle x, \mu^n \rangle \neq 0$ .

**Lemma 1.** *Let  $\mathbb{G}$  be a discrete quantum group, and let  $\mu \in \mathcal{P}(\mathbb{G})$  be irreducible. Then there exists  $k \in \mathbb{N}$  such that*

$$\lim_n \|\mu^{n+k} - \mu^n\| = 0.$$

*Proof.* Since  $\mu$  is irreducible, it follows from the definition that  $\mu_0 := \sum_{n=1}^\infty 2^{-n} \mu^n$  is a faithful normal state. Now, let  $\psi$  be the right Haar weight of  $\mathbb{G}$ , then we have

$$\psi(x \star \mu) 1 = (\psi \otimes \iota) \Gamma(x \star \mu) = ((\psi \otimes \iota) \Gamma(x)) \star \mu = \psi(x) 1$$

for all  $x \in l^\infty(\mathbb{G})^+$ . Since  $\psi$  is faithful, it follows that the map  $x \mapsto x \star \mu$  is faithful on  $l^\infty(\mathbb{G})$ . This, in particular, implies that  $\mu_0 \star \mu^m$  is faithful on  $l^\infty(\mathbb{G})$  for all  $m \in \mathbb{N}$ . Thus, for every positive  $x \in l^\infty(\mathbb{G})$  and  $n \in \mathbb{N}$ , there are  $n \leq m_1, m_2$  such that  $\langle \mu^{m_1}, x \rangle \neq 0$  and  $\langle \mu^{m_2}, x \rangle \neq 0$ . Let  $e \in l^\infty(\mathbb{G})$  be the central minimal projection obtained from the trivial representation of the dual compact quantum group, and suppose that  $n, k \in \mathbb{N}$  are such that

$$\langle \mu^n, e \rangle \neq 0 \quad \text{and} \quad \langle \mu^{n+k}, e \rangle \neq 0.$$

Then we have  $\lambda \delta_e \leq \mu^n$  and  $\lambda \delta_e \leq \mu^{n+k}$  for some positive  $\lambda \in \mathbb{R}$ , where  $\delta_e = e\psi$ . Hence, lemma follows from the non-commutative 0-2 law [4, Proposition 2.12].  $\square$

**Theorem 2.** *Let  $\mathbb{G}$  be a (infinite dimensional) discrete quantum group, and let  $\mu \in \mathcal{P}(\mathbb{G})$  be irreducible. Then*

$$(0.2) \quad \|x \star \mu^n\| \longrightarrow 0$$

for all  $x \in c_0(\mathbb{G})$ .

*Proof.* We first show that there exists a subnet  $(\mu^{n_i})_i$  such that

$$(0.3) \quad \lim_i \mu^{n_i+s} = 0$$

in the  $\sigma(l^1(\mathbb{G}), c_0(\mathbb{G}))$ -topology, for all  $s \in \mathbb{N}$ . So, let  $\mathcal{F}$  be a Banach limit, and define

$$\nu = \text{weak}^*\text{-}\lim_{\mathcal{F}} \mu^n \in l^1(\mathbb{G})^+.$$

Then  $\nu \star \mu = \mu \star \nu = \nu$ , and so

$$\mu \star (\nu \star x) = (\mu \star \nu) \star x = \nu \star x$$

for all  $x \in c_0(\mathbb{G})$ .

Now, let  $x \in c_0(\mathbb{G})^+$ , and suppose that  $\eta \in \mathcal{P}(\mathbb{G})$  is such that

$$\langle \nu \star x, \eta \rangle = \|\nu \star x\|.$$

Then we get

$$\begin{aligned} \langle (\|\nu \star x\| 1 - \nu \star x) \star \eta, \mu^n \rangle &= \langle \mu^n \star (\|\nu \star x\| 1 - \nu \star x), \eta \rangle \\ &= \langle \|\nu \star x\| 1 - \nu \star x, \eta \rangle \\ &= \|\nu \star x\| - \|\nu \star x\| = 0 \end{aligned}$$

for all  $n \in \mathbb{N}$ . Hence, it follows from the irreducibility of  $\mu$ , that

$$(\iota \otimes \eta) \Gamma(\|\nu \star x\| 1 - \nu \star x) = (\|\nu \star x\| 1 - \nu \star x) \star \eta = 0,$$

and therefore, we obtain

$$0 = \psi((\iota \otimes \eta) \Gamma(\|\nu \star x\| 1 - \nu \star x)) = \psi(\|\nu \star x\| 1 - \nu \star x),$$

where  $\psi$  is the right Haar weight. Since  $\|\nu \star x\| 1 - \nu \star x \geq 0$  and  $\psi$  is faithful, it follows that

$$\|\nu \star x\| 1 - \nu \star x = 0.$$

But since  $1 \notin c_0(\mathbb{G})$ , this shows that  $\nu \star x = 0$  for all  $x \in c_0(\mathbb{G})$ . From this, we see that  $\nu$  vanishes on

$$\{(\iota \otimes \omega) \Gamma(x) : \omega \in c_0(\mathbb{G})^*, x \in c_0(\mathbb{G})\},$$

and since the latter is dense in  $c_0(\mathbb{G})$ , we conclude that  $\nu = 0$ . Hence, for each  $0 \leq x \in c_0(\mathbb{G})$  we have

$$(0.4) \quad 0 \leq \liminf_n \langle x, \mu^n \rangle \leq \mathcal{F}(\langle x, \mu^n \rangle_{n=1}^\infty) = 0.$$

Now, let  $x_1, x_2, \dots, x_n$  be positive elements in  $c_0(\mathbb{G})$ , and define

$$x_0 = \sum_{k=1}^n x_k \in c_0(\mathbb{G})^+.$$

Then, by (0.4), we can choose a subnet  $(\mu^{n_j})$  of  $(\mu^n)$  such that  $\langle x_0, \mu^{n_j} \rangle \rightarrow 0$ . Thus, it follows that

$$\lim_j \langle x_k, \mu^{n_j} \rangle = 0.$$

for all  $1 \leq k \leq n$ . This shows that we can find a subnet  $(\mu^{n_i})$  of  $(\mu^n)$  such that  $\mu^{n_i} \rightarrow 0$  in the  $\sigma(l^1(\mathbb{G}), c_0(\mathbb{G}))$ -topology, whence (0.3) follows.

Now, towards a contradiction, suppose that there exists  $0 \leq x \in c_0(\mathbb{G})$  such that the limit in (0.2) does not converge to zero. Since the sequence  $\|x \star \mu^n\|$  is positive and non-increasing, it has a limit. So, there exists  $\alpha > 0$  such that  $\|x \star \mu^n\| \geq \alpha$  for all  $n \in \mathbb{N}$ , and therefore we can find  $\omega_n \in \mathcal{P}(\mathbb{G})$  such that

$$(0.5) \quad \langle x \star \mu^n, \omega_n \rangle \geq \alpha$$

for all  $n \in \mathbb{N}$ . Now, assume that

$$\rho = \lim_j \omega_{n_{i_j}} \star \mu^{n_{i_j}}$$

is a weak\* cluster point of  $\{\omega_{n_i} \star \mu^{n_i}\}$  in the unit ball of  $l^1(\mathbb{G})$ . Then (0.5) implies that  $\langle x, \rho \rangle \geq \alpha$ .

Moreover, by Lemma 1, there exists  $k \in \mathbb{N}$  such that

$$\begin{aligned} \langle a, \rho \star \mu^k - \rho \rangle &= \lim_j \langle a, \omega_{n_{i_j}} \star \mu^{n_{i_j}} \star \mu^k - \omega_{n_{i_j}} \star \mu^{n_{i_j}} \rangle \\ &\leq \lim_j \|a\| \|\mu^{k+n_{i_j}} - \mu^{n_{i_j}}\| = 0 \end{aligned}$$

for all  $a \in c_0(\mathbb{G})$ . Hence,

$$\rho = \rho \star \mu^k = \rho \star \mu^{nk}$$

for all  $n \in \mathbb{N}$ , which in particular, implies

$$(0.6) \quad \alpha \leq \langle x, \rho \rangle = \langle x, \rho \star \mu^{nk} \rangle$$

for all  $n \in \mathbb{N}$ . But, on the other hand, from (0.3) it follows that there exists  $m \in \mathbb{N}$  such that

$$\langle x, \rho \star \mu^{m+s} \rangle = \langle x \star \rho, \mu^{m+s} \rangle < \frac{\alpha}{2}$$

for all  $1 \leq s \leq k$ . Since  $k$  divides one of  $m+1, m+2, \dots, m+k$ , this contradicts (0.6), and therefore finishes the proof.  $\square$

**Remark 1.** The proof of Lemma 1 can be modified to prove the statement for the case of a locally compact quantum group  $\mathbb{G}$  whose von Neumann algebra  $\mathcal{L}^\infty(\mathbb{G})$  contains a non-zero central abelian projection, and  $\mu$  is a state on  $\mathcal{C}_0(\mathbb{G})$  with at least one non-singular convolution power. Then the same proof yields Theorem 2 in this case; in particular, this provides a new proof for the concentration functions problems in the classical setting, for spread-out probability measures on locally compact groups.

**Remark 2.** The convergence of convolution powers  $\mu^n$  to zero in the weak\* topology, in the classical case, holds for a more general class of probability measures  $\mu$ , namely those whose support generates  $G$  as a group, rather than a semigroup. But this is not the case for Theorem 2. For a counter-example, consider the additive group of integers  $(\mathbb{Z}, +)$ , and the probability measure  $\delta_1$ , the Dirac measure at 1. Then the group generated by the support of  $\delta_1$  is  $\mathbb{Z}$ , but every convolution power  $\delta_1^n = \delta_n$  induces an isometry on  $c_0(\mathbb{Z})$ .

## REFERENCES

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